A Discrete-Time Macroeconomic Model with a Financial Sector

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Introduction

- Brunnermeier and Sannikov (2012) argue that being able to solve for the full equilibrium dynamics of macro models is key in understanding the instability of the financial sector
- Most previous quantitative papers study approximate dynamics, typically via log-linearization
- BS solve a heterogeneous agent, **continuous-time**, GE model that displays:
  - fragility, stochastic volatility, and a time-varying real effect of financial shocks

Questions

- Does the BS model have a discrete time analog? Does the discrete time analog display the same properties?
  - yes
- Would similar existing models generate these properties if analyzed using better approximations/solution methods?
- Is there a missing element in previous models that excludes some of these properties?
  - maybe

Outline

1. Introduction
2. The discrete time model
3. Equilibrium characterization
4. Numerical solution
5. Simulations

2. The Discrete Time Model

- $t = 1, 2, \ldots$
- $v_t \in \{-1, 1\}$ is the random state/shock
- shocks are iid and $\pi = \Pr(v_t = 1) = \frac{1}{2}$

Goods and Agents

- Two goods: capital, k and consumption, c
- capital is used to produce the consumption good
- technology is linear in capital

$$y_t = ak_t$$

- Two agents: **households** (denoted by underscore variables) and **experts**

Experts and Households

- Both agents have linear preferences for consumption
- The households and experts subjective discount rates are, respectively, r and ρ
  - $r < \rho$, so experts are more impatient
- Notation: underlined variables denote household variables

$$\begin{array}{ll} \text{households} & \underline{c}_t, \underline{k}_t, \ldots \\ \text{experts} & c_t, k_t, \ldots \end{array}$$

Expert and Household Heterogeneity

- experts' consumption needs to be non-negative, but households can have negative consumption:

$$c_t \geq 0$$
$$\underline{c}_t \in \mathbb{R}$$

  - consumers have a "backyard technology"
- On average, expert-owned capital grows at rate g, while consumer-owned capital depreciates at rate δ

$$g > -\delta$$

Markets

- Agents trade capital at price q_t
  - k_t^c and $\underline{k}_t^c$ denote the capital *choices* of the agents
  - No capital short sales
- Agents also trade bonds, denoted by b, in zero net supply
  - Bonds pay an interest rate r_t (between t and $t+1$)
- Finally, agents can trade output for consumption

Timing of Events

- Start of t: capital k_t yields ak_t
- Within t: experts trade, repay debt, consume, and buy capital k_t^c
- End of t (or between t and $t+1$):

$$
\begin{aligned}
k_t^c &\longrightarrow k_{t+1} = (1+g)\, k_t^c + \sigma v_{t+1} k_t^c \\
\underline{k}_t^c &\longrightarrow \underline{k}_{t+1} = (1-\delta)\, \underline{k}_{t-1}^c + \sigma v_t \underline{k}_{t-1}^c ,
\end{aligned}
$$

  - σ is the volatility of the capital shock
  - capital shocks are aggregate shocks

Solvency Constraint

- Expert net worth (after production, but before trade) must always be non-negative:

$$n_t = (a + q_t) k_t + (1 + r_{t-1}) b_{t-1} \geq 0$$

- The solvency constraint can be interpreted as a collateral constraint with no default:

$$\begin{aligned} n_{t+1} &\geq 0 \\ -(1 + r_t) b_t &\leq \min_{v_{t+1}} \{(a + q_{t+1}(v_{t+1})) k_{t+1}(v_{t+1})\} \end{aligned}$$

  - note: $k_t = 0$ or $n_t = 0$ implies $c_t = c_{t+1} = ... = 0$

Definition of Equilibrium

Definition (Sequential Competitive Equilibrium)

A competitive equilibrium consists of prices (r_t, q_t) and allocations for experts (c_t, k_t^c, b_t) and households $(\underline{c}_t, \underline{k}_t^c, \underline{b}_t)$ such that:

1. Given prices, expert and household decisions are optimal
2. Markets Clear:

$$
\begin{aligned}
c_t + \underline{c}_t &= a k_t + a \underline{k}_t \\
k_t^c + \underline{k}_t^c &= k_t + \underline{k}_t \equiv K_t \\
b_t + \underline{b}_t &= 0
\end{aligned}
$$

3. The law of motion of aggregate capital is

$$
K_{t+1} = (1 + g + \sigma v_{t+1})\, k_t^c + (1 - \delta + \sigma v_{t+1})\, \underline{k}_t^c
$$

3. Equilibrium Characterization

- As in Brunnermeier and Sannikov (2012), we study an equilibrium in which the expert capital and bond choices are always *interior*
  - neither the solvency constraint nor the short-sale constraint ever binds for the expert
  - thus, due to linear utility, in equilibrium experts are indifferent to all bond and capital choices
  - however, there is a unique portfolio consistent with the equilibrium prices
- Even though the expert is always indifferent, there is a time-varying risk premium
  - the marginal utility of wealth fluctuates

3. Equilibrium Characterization

- Guess and verify that there is an equilibrium where experts always hold capital
- If experts borrow maximally they may be wiped out, which is inconsistent with the proposed equilibrium
- The equilibrium characterization suggests this equilibrium is unique if

$$\frac{(1+\rho)(g+\delta)}{\rho - r} > \sigma > \delta + g$$

Household Problem

$$\max_{\underline{c}_t, \underline{k}_t^c, \underline{b}_t} \sum_{t \geq 0} \frac{\mathbb{E}_0 \left[\underline{c}_t \right]}{(1+r)^t} \textit{ subject to}$$

$$\underline{c}_t + \underline{k}_t^c q_t + \underline{b}_t \leq (a + q_t) \underline{k}_t + (1 + r_{t-1}) \underline{b}_{t-1}$$

$$\underline{k}_t = (1 - \delta) \underline{k}_{t-1}^c + \sigma v_t \underline{k}_{t-1}^c$$

$$\underline{k}_t^c \geq 0$$

- Risk neutrality and $\underline{c}_t \in \mathbb{R}$ imply that markets in equilibrium

$$r_t = r$$

$$\mathbb{E}_t \left[\frac{(a + q_{t+1})(1 - \delta + \sigma v_{t+1})}{q_t} \right] \leq 1 + r$$

- with equality will if $\underline{k}_t > 0$.

Expert Problem

$$\max_{c_t, k_t^c, b_t} \sum_{t \geq 0} \frac{\mathbb{E}_0 \left[c_t \right]}{\left(1 + \rho \right)^t} \textit{ subject to}$$

$$c_t + k_t^c q_t + b_t \leq \left(a + q_t \right) k_t + \left(1 + r_{t-1} \right) b_{t-1} = n_t$$

$$k_t = \left(1 + g \right) k_{t-1}^c + \sigma v_t k_{t-1}^c$$

$$n_{t+1} \geq 0$$

$$c_t, k_t^c \geq 0$$

Recursive Expert Problem

As long as

$$\lim_{t\to\infty} V'(n_t)\, q_t = 0,$$

the expert solution coincides with the solution to the following Bellman equation:

$$V(n_t) = \max_{c_t, k_t^c, b_t} \left\{ c_t + \frac{1}{1+\rho} \mathbb{E}\left[V\left(n_{t+1}\left(v_{t+1}\right)\right)\right] \right\} \textit{ subject to}$$

$$c_t + k_t^c q_t + b_t \le (a + q_t)\, k_t + (1 + r_{t-1})\, b_{t-1} = n_t$$

$$k_t = (1+g)\, k_{t-1}^c + \sigma v_t k_{t-1}^c$$

$$n_{t+1} \ge 0$$

$$c_t, k_t^c \ge 0$$

Expert Problem

Lemma

The expert's value function is linear in net worth

Proof.

Suppose two experts, A and B, have value functions V_t^A and V_t^B and net worths n_t^A and n_t^B. Let $\left(c_t^A, k_t^{c,A}, b_t^A\right)$ be $A's$ optimal plan, and let $\zeta = n_t^B / n_t^A$. Because the budget set is homogeneous of degree 1 in the choice variables, $\left(\zeta c_t^A, \zeta k_t^{c,A}, \zeta b_t^A\right)$ is a feasible plan for B and delivers utility ζV_t^B. Therefore, $V_t^A \geq \zeta V_t^B$. Reversing the argument, we find $V_t^B \geq \frac{1}{\zeta} V_t^A$. It follows that

$$\frac{V_t^B}{n_t^B} = \frac{V_t^A}{n_t^A}$$

□

Expert Problem

This lemma implies that we can write the recursive problem as

$$\theta_t n_t = \max_{c_t, k_t^c, b_t} \left\{ c_t + \frac{1}{1+\rho} \mathbb{E}\left[\theta_{t+1}\left(v_{t+1}\right) n_{t+1}\left(v_{t+1}\right)\right] \right\} \textit{ subject to}$$

$$c_t + k_t^c q_t + b_t \leq \left(a + q_t\right) k_t + \left(1 + r_{t-1}\right) b_{t-1} = n_t$$

$$k_t = \left(1+g\right) k_{t-1}^c + \sigma v_t k_{t-1}^c$$

$$n_{t+1} \geq 0$$

$$c_t, k_t^c \geq 0,$$

where θ_t is the marginal utility of wealth.

Expert Problem

Assuming experts always hold some capital and never borrow maximally, first order conditions imply:

1. $$\theta_t \geq 1$$

    - with equality when $c_t > 0$

2. $$\theta_t q_t = \frac{1}{1+\rho}\mathbb{E}\left[\theta_{t+1}\left(a + q_{t+1}\right)\left(1 + g + \sigma v_{t+1}\right)\right]$$

3. $$\frac{\theta_t}{1+r} = \frac{1}{1+\rho}\mathbb{E}\left[\theta_{t+1}\right]$$

Characterization of the Equilibrium: State Variables

- The **state variables** for the recursive equilibrium are the distribution of net worth and aggregate capital:

$$n_t, \underline{n}_t, K_t$$

- However, due to the backyard technology, $\underline{n}_t$ is irrelevant for prices
- Therefore, two variables are sufficient:

$$n_t, K_t$$

Scale Invariance

Theorem (Scale Invariance)

If (r_t, q_t) is an equilibrium of an economy with aggregate capital K_t and expert net worth n_t, then it is also an equilibrium of an economy with capital ζK_t and net worth ζn_t, for any $\zeta > 0$.

- Scale invariance implies that one state variable is sufficient, the ratio of expert net worth to aggregate capital:

$$\eta_t = \frac{n_t}{K_t}$$

Equilibrium Equations

The equilibrium consists of η_t ranges $(0, \overline{\eta})$ and $[\overline{\eta}, \eta^*]$ and functions $q(\eta_t)$, $\theta(\eta_t)$, and $\psi(\eta_t)$ such that

1. Law of Motion of the State Variable:

$$\eta_{t+1}(v_{t+1}) = \min\left\{F\left(v_{t+1}, q\left(v_{t+1}\right); q_t, \psi_t, \eta_t\right), \eta^*\right\}$$

   - $c_{t+1} > 0$ if $F\left(v_{t+1}, q\left(v_{t+1}\right); q_t, \psi_t, \eta_t\right) > \eta^*$ and $c_{t+1} = 0$ otherwise
   - when $\overline{\eta} \leq \eta_t < \eta^*$, the expert is *just* able buy all of the capital

2. Household optimality:

$$(1 - \psi_t)\left(\mathbb{E}_t\left[(a + q_{t+1})(1 - \delta + \sigma v_{t+1})\right] - q_t(1 + r)\right) = 0$$

   - $0 < \psi(\eta_t) < 1$ if $\eta_t \in (0, \overline{\eta})$ and $\psi(\eta_t) = 1$ if $\eta_t \geq \overline{\eta}$

Equilibrium Equations

3. Expert Capital FOC:

$$\theta_t q_t = \frac{1}{1+\rho} \mathbb{E}\left[\theta_{t+1}\left(a + q_{t+1}\right)\left(1 + g + \sigma v_{t+1}\right)\right]$$

4. Expert Debt FOC:

$$\frac{\theta_t}{1+r} = \frac{1}{1+\rho} \mathbb{E}\left[\theta_{t+1}\right]$$

- $\theta\left(\eta_{t+1}\right) = 1$ if $F\left(v_{t+1}, q\left(v_{t+1}\right); q_t, \psi_t, \eta_t\right) \geq \eta^*$ and $\theta\left(\eta_{t+1}\right) > 1$ otherwise

Equilibrium Equations

5. Boundary Conditions:
    - $\eta = 0$

$$q(0) = \underline{q} = \frac{a(1-\delta)}{r+\delta}$$
$$\theta(0) = \infty$$

    - $\eta = \eta^*$

$$q'(\eta^*) = 0$$
$$\theta(\eta^*) = 1$$
$$\theta'(\eta^*) = 0$$

- Note that we have 5 boundary conditions for a system of second order difference equations with an unknown boundary

4. Numerical Solution

Shooting Method ("from the right")

- The unknowns are the functions q and θ and the initial conditions η^* and $q(\eta^*)$.
  - however, if we knew η^* and $q(\eta^*)$, then we could calculate all variables in the next period following a bad shock (-1)
  - with knowledge of all variables in the next period, we can again advance the system with another bad shock
- Eventually, the initial guesses η^* and $q(\eta^*)$ yield values for $q(0)$ and $\theta(0)$ (or the system produces nonsense before $\eta = 0$)
- In any case, we iterate on η^* and $q(\eta^*)$ until the function paths are consistent with the boundary conditions

Solution

Why can we always advance the system?

- Suppose we observe a sequence of bad shocks and the corresponding variables of interest:

$$(\eta_0, \psi_0, \theta_0, q_0), \ldots, (\eta_t, \psi_t, \theta_t, q_t)$$

- We can calculate the $v_{t+1} = 1$ variables via interpolation because $\eta_{t+1}(v_{t+1} = 1) \in [\eta_t, \eta_0]$.
  - approximation error
- Finally, because v_{t+1} only takes two values, the FOCs precisely pin down $\eta_{t+1}(v_{t+1} = -1)$ and all other variables following a bad shock!

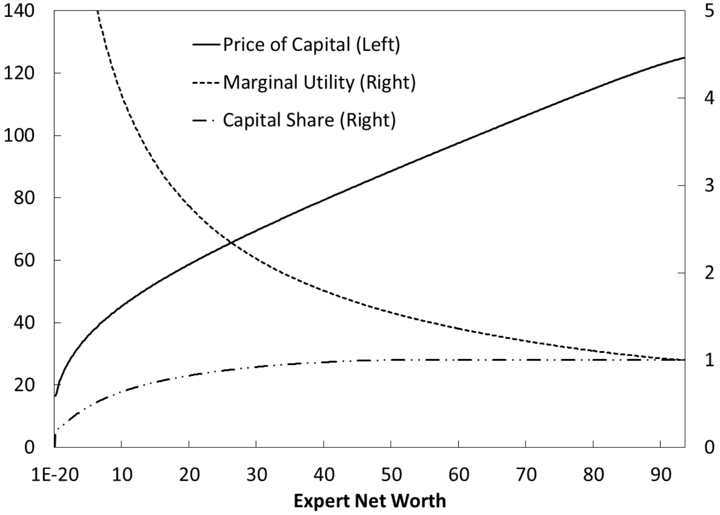
5. Simulation

- Recall we need $\frac{(1+\rho)(g+\delta)}{\rho - r} > \sigma > \delta + g$
- Example:
  - $r = .05$
  - $\rho = .06$
  - $a = 1$
  - $\sigma = .07$
  - $\delta = .01$
  - $g = .05$

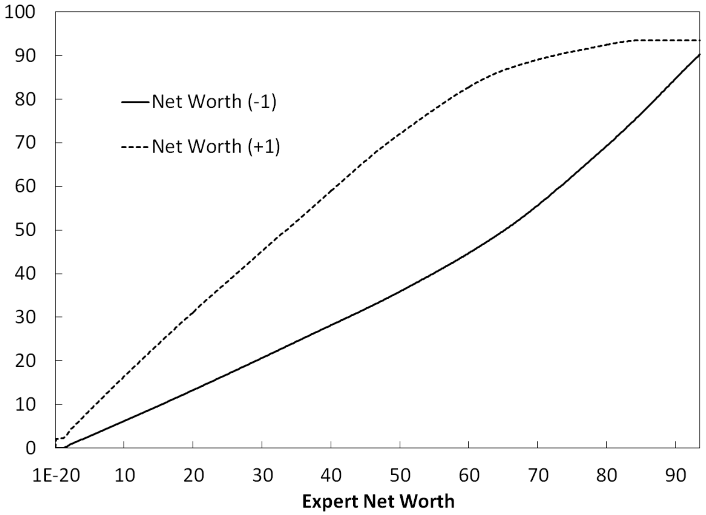
$$\Longrightarrow$$

$$(\eta^*, q(\eta^*)) = (93.52, 124.96)$$

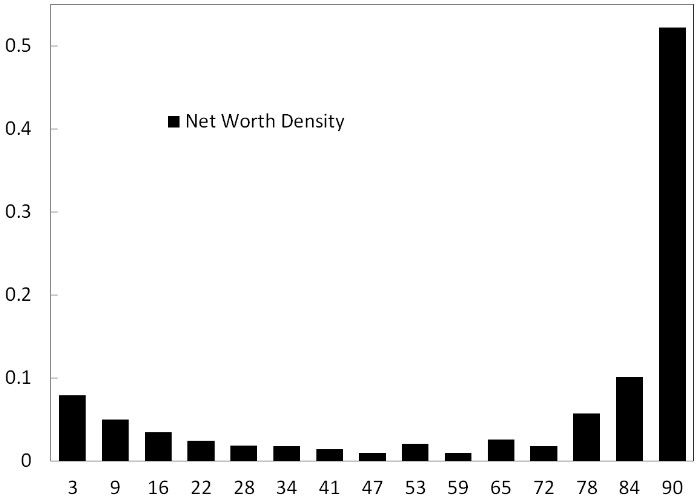
Theta(eta), q(eta), and psi(eta)



Eta(1) and Eta(-1)



Eta Density (via simulation)



Conclusion

- The model of Brunnermeier and Sannikov (2012) has a clear discrete time analog
- Using the methods and equilibrium form of Brunnermeier and Sannikov (2012), we solve away from steady-state a heterogeneous agent macro model with financial frictions
  - It is risk-neutrality and the interior form of the equilibrium that facilitate the solution
  - Discrete vs. continuous is irrelevant
- For our parameters, the discrete model also generates fragility and stochastic volatility

Characterization of the Equilibrium: State Variables

- The state variable η_t evolves according to

$$\eta_{t+1}\left(v_{t+1}\right) = \frac{1+r}{\Psi_{t+1}\left(v_{t+1}\right)}\eta_t + \frac{R^k_{t+1}\left(v_{t+1}\right)-(1+r)}{\Psi_{t+1}\left(v_{t+1}\right)}q_t\psi_t$$
$$= F\left(v_{t+1}, q_{t+1}\left(v_{t+1}\right); q_t, \psi_t, \eta_t\right)$$

  where we define

  - post-trade capital distribution:

$$\psi_t = \frac{k^c_t}{K_t}$$

  - return on capital:

$$R^k_{t+1}\left(v_{t+1}\right) = \frac{a+q_{t+1}\left(v_{t+1}\right)}{q_t}\left(1+g+\sigma v_{t+1}\right)$$

  - growth rate of capital:

$$\Psi_{t+1}\left(v_{t+1}\right) = 1 + g\psi_t - \delta\left(1-\psi_t\right) + \sigma v_{t+1}$$