A Discrete-Time Macroeconomic Model with a Financial Sector

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Brunnermeier and Sannikov (2012) argue that being able to solve for the full equilibrium dynamics of macro models is key in understanding the instability of the financial sector.

Most previous quantitative papers study approximate dynamics, typically via log-linearization.

BS solve a heterogeneous agent, continuous-time, GE model that displays:

- fragility, stochastic volatility, and a time-varying real effect of financial shocks.
Questions

- Does the BS model have a discrete time analog? Does the discrete time analog display the same properties?
  - yes

- Would similar existing models generate these properties if analyzed using better approximations/solution methods?

- Is there a missing element in previous models that excludes some of these properties?
  - maybe
Outline

1. Introduction
2. The discrete time model
3. Equilibrium characterization
4. Numerical solution
5. Simulations
2. The Discrete Time Model

- $t = 1, 2, ...$
- $v_t \in \{-1, 1\}$ is the random state/shock
- shocks are iid and $\pi = \Pr(v_t = 1) = \frac{1}{2}$
Goods and Agents

- Two goods: capital, $k$ and consumption, $c$
- capital is used to produce the consumption good
- technology is linear in capital
  \[ y_t = a k_t \]
- Two agents: households (denoted by underscore variables) and experts
Experts and Households

- Both agents have linear preferences for consumption.
- The households and experts' subjective discount rates are, respectively, $r$ and $\rho$.
  - $r < \rho$, so experts are more impatient.
- Notation: underlined variables denote household variables.

  households $\underline{c}_t, \underline{k}_t, \ldots$

  experts $c_t, k_t, \ldots$
Expert and Household Heterogeneity

- experts’ consumption needs to be non-negative, but households can have negative consumption:

\[ c_t \geq 0 \]
\[ c_t \in \mathbb{R} \]

- consumers have a “backyard technology”

- On average, expert-owned capital grows at rate \( g \), while consumer-owned capital depreciates at rate \( \delta \)

\[ g > -\delta \]
Markets

- Agents trade capital at price $q_t$
  - $k^C_t$ and $k^C_{t-1}$ denote the capital choices of the agents
  - No capital short sales
- Agents also trade bonds, denoted by $b$, in zero net supply
  - Bonds pay an interest rate $r_t$ (between $t$ and $t + 1$)
- Finally, agents can trade output for consumption
Timing of Events

- Start of $t$: capital $k_t$ yields $ak_t$
- Within $t$: experts trade, repay debt, consume, and buy capital $k^c_t$
- End of $t$ (or between $t$ and $t+1$):

$$k^c_t \rightarrow k_{t+1} = (1 + g)k^c_t + \sigma v_{t+1} k^c_t$$
$$\underline{k}^c_t \rightarrow \underline{k}_{t+1} = (1 - \delta)\underline{k}^c_{t-1} + \sigma v_t \underline{k}^c_{t-1},$$

- $\sigma$ is the volatility of the capital shock
- capital shocks are aggregate shocks
Solvency Constraint

- Expert net worth (after production, but before trade) must always be non-negative:

\[ n_t = (a + q_t) k_t + (1 + r_{t-1}) b_{t-1} \geq 0 \]

- The solvency constraint can be interpreted as a collateral constraint with no default:

\[ n_{t+1} \geq 0 \]
\[ - (1 + r_t) b_t \leq \min_{v_{t+1}} \{(a + q_{t+1} (v_{t+1})) k_{t+1} (v_{t+1})\} \]

- note: \( k_t = 0 \) or \( n_t = 0 \) implies \( c_t = c_{t+1} = \ldots = 0 \)
Definition of Equilibrium

Definition (Sequential Competitive Equilibrium)
A competitive equilibrium consists of prices \((r_t, q_t)\) and allocations for experts \((c_t, k_t^c, b_t)\) and households \((c_t, k_t^c, b_t)\) such that:

1. Given prices, expert and household decisions are optimal
2. Markets Clear:

\[
\begin{align*}
    c_t + c_t &= a k_t + a k_t \\
    k_t^c + k_t^c &= k_t + k_t \equiv K_t \\
    b_t + b_t &= 0
\end{align*}
\]

3. The law of motion of aggregate capital is

\[
K_{t+1} = (1 + g + \sigma v_{t+1}) k_t^c + (1 - \delta + \sigma v_{t+1}) k_t^c
\]
3. Equilibrium Characterization

- As in Brunnermeier and Sannikov (2012), we study an equilibrium in which the expert capital and bond choices are always *interior*
  
  - neither the solvency constraint nor the short-sale constraint ever binds for the expert
  - thus, due to linear utility, in equilibrium experts are indifferent to all bond and capital choices
  - however, there is a unique portfolio consistent with the equilibrium prices

- Even though the expert is always indifferent, there is a time-varying risk premium
  
  - the marginal utility of wealth fluctuates
3. Equilibrium Characterization

- Guess and verify that there is an equilibrium where experts always hold capital.
- If experts borrow maximally they may be wiped out, which is inconsistent with the proposed equilibrium.
- The equilibrium characterization suggests this equilibrium is unique if

\[
\frac{(1 + \rho)(g + \delta)}{\rho - r} > \sigma > \delta + g
\]
Household Problem

\[
\max_{c_t, k_t^c, b_t} \sum_{t \geq 0} \frac{E_0 [c_t]}{(1 + r)^t} \hspace{1cm} \text{subject to}
\]

\[
c_t + k_t^c q_t + b_t \leq (a + q_t) k_t + (1 + r_{t-1}) b_{t-1}
\]

\[
k_t = (1 - \delta) k_{t-1}^c + \sigma v_t k_{t-1}^c
\]

\[
k_t^c \geq 0
\]

- Risk neutrality and \( c_t \in \mathbb{R} \) imply that markets in equilibrium

\[
r_t = r
\]

\[
E_t \left[ \frac{(a + q_{t+1}) (1 - \delta + \sigma v_{t+1})}{q_t} \right] \leq 1 + r
\]

- with equality will if \( k_t > 0 \).
Expert Problem

\[ \max_{c_t, k_t^c, b_t} \sum_{t \geq 0} \frac{\mathbb{E}_0 [c_t]}{(1 + \rho)^t} \text{ subject to } \]
\[ c_t + k_t^c q_t + b_t \leq (a + q_t) k_t + (1 + r_{t-1}) b_{t-1} = n_t \]
\[ k_t = (1 + g) k_{t-1}^c + \sigma v_t k_{t-1}^c \]
\[ n_{t+1} \geq 0 \]
\[ c_t, k_t^c \geq 0 \]
Recursive Expert Problem

As long as

\[ \lim_{t \to \infty} V' (n_t) q_t = 0, \]

the expert solution coincides with the solution to the following Bellman equation:

\[
V (n_t) = \max_{c_t, k_t^c, b_t} \left\{ c_t + \frac{1}{1 + \rho} \mathbb{E} [V (n_{t+1} (v_{t+1}))] \right\} \quad \text{subject to}
\]

\[
c_t + k_t^c q_t + b_t \leq (a + q_t) k_t + (1 + r_{t-1}) b_{t-1} = n_t
\]

\[
k_t = (1 + g) k_{t-1}^c + \sigma v_t k_{t-1}^c
\]

\[
n_{t+1} \geq 0
\]

\[
c_t, k_t^c \geq 0
\]
Expert Problem

Lemma
The expert’s value function is linear in net worth

Proof.
Suppose two experts, A and B, have value functions $V_t^A$ and $V_t^B$ and net worths $n_t^A$ and $n_t^B$. Let $\left(c_t^A, k_t^A, b_t^A\right)$ be A’s optimal plan, and let $\zeta = n_t^B / n_t^A$. Because the budget set is homogeneous of degree 1 in the choice variables, $\left(\zeta c_t^A, \zeta k_t^A, \zeta b_t^A\right)$ is a feasible plan for B and delivers utility $\zeta V_t^B$. Therefore, $V_t^A \geq \zeta V_t^B$. Reversing the argument, we find $V_t^B \geq \frac{1}{\zeta} V_t^A$. It follows that

$$\frac{V_t^B}{n_t^B} = \frac{V_t^A}{n_t^A}$$
Expert Problem

This lemma implies that we can write the recursive problem as

\[ \theta_t n_t = \max_{c_t, k^c_t, b_t} \left\{ c_t + \frac{1}{1 + \rho} \mathbb{E} \left[ \theta_{t+1} (v_{t+1}) n_{t+1} (v_{t+1}) \right] \right\} \quad \text{subject to} \]
\[ c_t + k^c_t q_t + b_t \leq (a + q_t) k_t + (1 + r_{t-1}) b_{t-1} = n_t \]
\[ k_t = (1 + g) k^c_{t-1} + \sigma v_t k^c_{t-1} \]
\[ n_{t+1} \geq 0 \]
\[ c_t, k^c_t \geq 0, \]

where \( \theta_t \) is the marginal utility of wealth.
Expert Problem

Assuming experts always hold some capital and never borrow maximally, first order conditions imply:

1. \[ \theta_t \geq 1 \]
   
   - with equality when \( c_t > 0 \)

2. \[ \theta_t q_t = \frac{1}{1 + \rho} \mathbb{E} \left[ \theta_{t+1} (a + q_{t+1}) (1 + g + \sigma v_{t+1}) \right] \]

3. \[ \frac{\theta_t}{1 + r} = \frac{1}{1 + \rho} \mathbb{E} [\theta_{t+1}] \]
Characterization of the Equilibrium: State Variables

- The **state variables** for the recursive equilibrium are the distribution of net worth and aggregate capital:

  \[ n_t, n_{t}, K_t \]

- However, due to the backyard technology, \( n_{t} \) is irrelevant for prices
- Therefore, two variables are sufficient:

  \[ n_t, K_t \]
Scale Invariance

Theorem (Scale Invariance)

If \((r_t, q_t)\) is an equilibrium of an economy with aggregate capital \(K_t\) and expert net worth \(n_t\), then it is also an equilibrium of an economy with capital \(\zeta K_t\) and net worth \(\zeta n_t\), for any \(\zeta > 0\).

- Scale invariance implies that one state variable is sufficient, the ratio of expert net worth to aggregate capital:

\[ \eta_t = \frac{n_t}{K_t} \]
Equilibrium Equations

The equilibrium consists of \( \eta_t \) ranges \((0, \bar{\eta})\) and \([\bar{\eta}, \eta^*]\) and functions \(q(\eta_t), \theta(\eta_t),\) and \(\psi(\eta_t)\) such that

1. Law of Motion of the State Variable:

\[
\eta_{t+1} (\nu_{t+1}) = \min \left\{ F (\nu_{t+1}, q (\nu_{t+1}); q_t, \psi_t, \eta_t), \eta^* \right\}
\]

- \(c_{t+1} > 0\) if \(F (\nu_{t+1}, q (\nu_{t+1}); q_t, \psi_t, \eta_t) > \eta^* \) and \(c_{t+1} = 0\) otherwise
- when \(\bar{\eta} \leq \eta_t < \eta^*\), the expert is just able buy all of the capital

2. Household optimality:

\[
(1 - \psi_t) (I_t [(a + q_{t+1}) (1 - \delta + \sigma\nu_{t+1})] - q_t (1 + r)) = 0
\]

- \(0 < \psi (\eta_t) < 1\) if \(\eta_t \in (0, \bar{\eta})\) and \(\psi (\eta_t) = 1\) if \(\eta_t \geq \bar{\eta}\)
Equilibrium Equations

3. Expert Capital FOC:

\[ \theta_t q_t = \frac{1}{1 + \rho} \mathbb{E} [\theta_{t+1} (a + q_{t+1}) (1 + g + \sigma v_{t+1})] \]

4. Expert Debt FOC:

\[ \frac{\theta_t}{1 + r} = \frac{1}{1 + \rho} \mathbb{E} [\theta_{t+1}] \]

- \[ \theta(\eta_{t+1}) = 1 \text{ if } F(v_{t+1}, q(v_{t+1}); q_t, \psi_t, \eta_t) \geq \eta^* \text{ and } \theta(\eta_{t+1}) > 1 \text{ otherwise} \]
Equilibrium Equations

5. Boundary Conditions:
   - $\eta = 0$

   $q(0) = q = \frac{a(1 - \delta)}{r + \delta}$
   $\theta(0) = \infty$

   - $\eta = \eta^*$

   $q'(\eta^*) = 0$
   $\theta(\eta^*) = 1$
   $\theta'(\eta^*) = 0$

   Note that we have 5 boundary conditions for a system of second order difference equations with an unknown boundary.
4. Numerical Solution

Shooting Method ("from the right")

- The unknowns are the functions \( q \) and \( \theta \) and the initial conditions \( \eta^* \) and \( q(\eta^*) \).
  - however, if we knew \( \eta^* \) and \( q(\eta^*) \), then we could calculate all variables in the next period following a bad shock (−1)
  - with knowledge of all variables in the next period, we can again advance the system with another bad shock

- Eventually, the initial guesses \( \eta^* \) and \( q(\eta^*) \) yield values for \( q(0) \) and \( \theta(0) \) (or the system produces nonsense before \( \eta = 0 \))

- In any case, we iterate on \( \eta^* \) and \( q(\eta^*) \) until the function paths are consistent with the boundary conditions
Why can we always advance the system?

- Suppose we observe a sequence of bad shocks and the corresponding variables of interest:
  \[(\eta_0, \psi_0, \theta_0, q_0), \ldots, (\eta_t, \psi_t, \theta_t, q_t)\]

- We can calculate the \(v_{t+1} = 1\) variables via interpolation because \(\eta_{t+1} (v_{t+1} = 1) \in [\eta_t, \eta_0]\).
  - approximation error

- Finally, because \(v_{t+1}\) only takes two values, the FOCs precisely pin down \(\eta_{t+1} (v_{t+1} = -1)\) and all other variables following a bad shock!
5. Simulation

- Recall we need \( \frac{(1+\rho)(g+\delta)}{\rho-r} > \sigma > \delta + g \)

- Example:
  - \( r = .05 \)
  - \( \rho = .06 \)
  - \( a = 1 \)
  - \( \sigma = .07 \)
  - \( \delta = .01 \)
  - \( g = .05 \)

\[
\implies (\eta^*, q(\eta^*)) = (93.52, 124.96)
\]
Theta(\(\eta\)), q(\(\eta\)), and \(\psi(\eta)\)
Eta(1) and Eta(-1)

Graph showing the relationship between Expert Net Worth and Net Worth for values of -1 and +1.
Conclusion

- The model of Brunnermeier and Sannikov (2012) has a clear discrete time analog.
- Using the methods and equilibrium form of Brunnermeier and Sannikov (2012), we solve away from steady-state a heterogeneous agent macro model with financial frictions.
  - It is risk-neutrality and the interior form of the equilibrium that facilitate the solution.
  - Discrete vs. continuous is irrelevant.
- For our parameters, the discrete model also generates fragility and stochastic volatility.
Characterization of the Equilibrium: State Variables

- The state variable $\eta_t$ evolves according to

\[
\eta_{t+1} (v_{t+1}) = \frac{1 + r}{\Psi_{t+1} (v_{t+1})} \eta_t + \frac{R_{t+1}^k (v_{t+1}) - (1 + r)}{\Psi_{t+1} (v_{t+1})} q_t \psi_t
\]

\[
= F (v_{t+1}, q_{t+1} (v_{t+1}); q_t, \psi_t, \eta_t)
\]

where we define

- post-trade capital distribution:

\[
\psi_t = \frac{k_t^c}{K_t}
\]

- return on capital:

\[
R_{t+1}^k (v_{t+1}) = \frac{a + q_{t+1} (v_{t+1})}{q_t} (1 + g + \sigma v_{t+1})
\]

- growth rate of capital:

\[
\Psi_{t+1} (v_{t+1}) = 1 + g \psi_t - \delta (1 - \psi_t) + \sigma v_{t+1}
\]